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The pseudospin symmetric solution of the Morse potential for any κ state

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Abstract
In this paper, the energy eigenvalues and the corresponding spinors of the Dirac particles are obtained and the pseudospin symmetric solution of the attractive scalar and repulsive vector Morse potential for any spin–orbit quantum number κ is provided by using the Pekeris approximation within the framework of the asymptotic iteration method.

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1. Introduction

The pseudospin symmetry (PSS) with the nuclear shell model was introduced many years ago [1, 2], and it has been widely used to explain a number of phenomena in nuclear physics and related areas (see [3] and reference therein). It has been shown that the quasi-degenerate pseudospin doublets in nuclei arise from the near equality of the magnitude of the attractive scalar and repulsive vector potentials, i.e., $V_S \sim V_V$ [3, 4]. PSS is seen as a quasi-degeneracy of the doublet single-particle states and characterized with the quantum numbers $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n, l and j are the single-particle radial, orbital and total angular momentum quantum numbers, respectively. The total angular momentum is written by a pseudo-angular momentum $\tilde{\ell} = \ell + 1$ and a pseudospin $\tilde{s} = 1/2$ as $j = \tilde{\ell} + \tilde{s}$ [5]. Pseudospin symmetry and spin symmetry [4, 5] occur for $\Sigma(r) = C = \text{const}$ and $\Delta = C = \text{const}$ in the Dirac equation, respectively. Pseudospin and spin symmetry have been observed in several nuclei for a few potentials such as the harmonic oscillator [6–10], Morse [11, 12] and Wood–Saxon [13, 14].

Recently, for any spin–orbit quantum number κ, the quasi-analytical solution of the Dirac equation has been presented for the Morse potential [15] with pseudospin symmetry using the Nifikorov–Uvarov [11] and the exact quantization rule methods [12]. We should point out here, however, that the conclusions of both papers regarding the bound state solutions of the exact pseudospin symmetry Morse potential for the Dirac particles are inconsistent.

Therefore, in this paper, our aim is to solve the Morse potential by using a different and more practical method called the asymptotic iteration method (AIM) [16] within the Pekeris approximation and to obtain the relativistic bound state energy eigenvalues and the corresponding Dirac spinors within the pseudospin symmetry concept. In the next section, we present the AIM with all necessary formulae to perform our calculations. In section 3.1, a brief introduction to the usual Dirac formalism are presented. In section 3.2, we present the Pekeris approximation, and in section 3.3 we investigate the relativistic bound state eigenvalues and the corresponding spinors of Dirac particles for $\kappa \neq 0$ with the pseudospin symmetric case $\Sigma(r) = C = \text{const}$. Finally, section 4 is devoted to our summary and conclusion.

2. The asymptotic iteration method

The AIM [16] is proposed to solve the second-order differential equations of the form

$$y_n''(x) = \lambda_0(x) y_n'(x) + s_0(x) y_n(x), \tag{1}$$

where $\lambda_0(x) \neq 0$ and the prime denotes the derivative with respect to x. The variables, $s_0(x)$ and $\lambda_0(x)$, are sufficiently differentiable. To find a general solution to this equation, we differentiate equation (1) with respect to x; we find

$$y_n'''(x) = \lambda_1(x) y_n'(x) + s_1(x) y_n(x), \tag{2}$$

where

$$\begin{aligned} \lambda_1(x) &= \lambda_0'(x) + s_0(x) + \lambda_0^2(x), \\ s_1(x) &= s_0'(x) + s_0(x)\lambda_0(x). \end{aligned} \tag{3}$$

Similarly, the second derivative of equation (1) yields

$$y_n^{(4)}(x) = \lambda_2(x) y_n'(x) + s_2(x) y_n(x), \tag{4}$$

where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x), \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \tag{5}$$

Equation (1) can be easily iterated up to the $(k+1)$th and $(k+2)$th derivatives, $k = 1, 2, 3, \ldots$. Therefore, we have

$$\begin{aligned} y_n^{(k+1)}(x) &= \lambda_{k-1}(x) y_n'(x) + s_{k-1}(x) y_n(x), \\ y_n^{(k+2)}(x) &= \lambda_k(x) y_n'(x) + s_k(x) y_n(x), \end{aligned} \tag{6}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x), \end{aligned} \tag{7}$$

which are called the recurrence relations. From the ratio of the $(k+2)$th and $(k+1)$th derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln\left[y_n^{(k+1)}(x)\right] = \frac{y_n^{(k+2)}(x)}{y_n^{(k+1)}(x)} = \frac{\lambda_k(x)\left[y_n'(x) + \frac{s_k(x)}{\lambda_k(x)} y_n(x)\right]}{\lambda_{k-1}(x)\left[y_n'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} y_n(x)\right]}. \tag{8}$$

For sufficiently large k, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \tag{9}$$

which is the 'asymptotic' aspect of the method, then, equation (8) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[y_n^{(k+1)}(x)\right] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{10}$$

which yields

$$y_n^{(k+1)}(x) = C_1 \exp\left(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)}\,\mathrm{d}x\right) = C_1\lambda_{k-1}(x)\exp\left(\int [\alpha(x)+\lambda_0(x)]\,\mathrm{d}x\right), \tag{11}$$

where C_1 is the integration constant. By inserting equation (11) into equation (6), the first-order differential equation is obtained as

$$y_n'(x) + \alpha(x)y_n(x) = C_1 \exp\left(\int [\alpha(x)+\lambda_0(x)]\,\mathrm{d}x\right). \tag{12}$$

This first-order differential equation can easily be solved and the general solution of equation (1) can be obtained as

$$y_n(x) = \exp\left(-\int^x \alpha(x_1)\,\mathrm{d}x_1\right)\left[C_2 + C_1\int^x \exp\left(\int^{x_1} [\lambda_0(x_2)+2\alpha(x_2)]\,\mathrm{d}x_2\right)\mathrm{d}x_1\right]. \tag{13}$$

For a given potential, the Dirac equation is converted to the form of equation (1). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and the $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given in equation (7). The termination condition of the method in equation (9) can be arranged as

$$\delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0, \qquad k = 1, 2, 3, \ldots \tag{14}$$

where k shows the iteration number. For the exactly solvable potentials, the energy eigenvalues are obtained from the roots of equation (14) and the radial quantum number n is equal to the iteration number k for this case. For nontrivial potentials that have no exact solutions, for a specific n principal quantum number, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wavefunction or the minimum value of the potential [16–18] and the approximate energy eigenvalues are obtained by iteration from the roots of equation (14) for sufficiently great values of k. We should point out that k is always greater than n in these numerical solutions.

The general solution of equation (1) is given by equation (13). The first part of equation (13) gives us polynomial solutions that are convergent and physical, whereas the second part of equation (13) gives us non-physical solutions that are divergent. Although equation (13) is the general solution of equation (1), we take the coefficient of the second part (C_1) as zero in order to find the square integrable solutions. Therefore, the corresponding eigenfunctions can be derived from the following wavefunction generator for exactly solvable potentials:

$$y_n(x) = C_2 \exp\left(-\int^x \frac{s_n(x_1)}{\lambda_n(x_1)}\,\mathrm{d}x_1\right), \tag{15}$$

where n represents the radial quantum number.

3. Exact analytical solution

3.1. Dirac equation

The stationary state Dirac wave equation [5, 10, 13, 14, 19] for a single particle with mass M in a scalar $S(\vec{r})$ and a vector potential $V(\vec{r})$ can be given as in unit $\hbar = c = 1$:

$$[\vec{\alpha}\cdot\vec{p} + \beta(M + S(\vec{r})) + V(\vec{r})]\psi(\vec{r}) = E\psi(\vec{r}), \tag{16}$$

where $\vec{p}$ and E are the momentum operator and the total relativistic energy of the system, respectively. $\vec{\alpha}$ and β are 4×4 Dirac matrices, i.e,

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma_i} \\ \vec{\sigma_i} & 0 \end{pmatrix}, \qquad \text{and} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{17}$$

where I is the 2×2 unit matrix and $\vec{\sigma}_{i=x,y,z}$ are 2×2 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{18}$$

The Dirac spinors may be written according to the upper (large) $f_{n\kappa}$ and lower (small) $g_{n\kappa}$ components as follows:

$$\psi_{n\kappa}(\vec{r}) = \begin{pmatrix} f_{n\kappa} \\ g_{n\kappa} \end{pmatrix} = \begin{pmatrix} \frac{F_{n\kappa(r)}}{r} Y^{\ell}_{jm}(\theta, \phi) \\ \frac{iG_{n\kappa(r)}}{r} Y^{\tilde{\ell}}_{jm}(\theta, \phi) \end{pmatrix}, \tag{19}$$

where $Y^{\ell}_{jm}(\theta, \phi)$ and $Y^{\tilde{\ell}}_{jm}(\theta, \phi)$ are the spin and pseudospin spherical harmonics, n is the radial quantum number and m is the projection of the angular momentum on the z axis. The $\hat{H}$, $\hat{L}$ and $\hat{J}$ operators do not create a complete set and they do not have the same Dirac spinors since the orbital angular momentum $\hat{L}$ cannot commute with the Dirac Hamiltonian. The spin–orbit matrix operator $\hat{K}$ is defined as $\hat{K} = -\beta(\vec{\sigma} \cdot \vec{L} + 1)$. The eigenvalues of $\hat{K}$ are $\kappa = (j + 1/2)$, where $\kappa = -(j + 1/2) < 0$ is for the aligned spin $j = l + 1/2$ ($s_{1/2}$, $p_{3/2}$, etc) and $\kappa = (j + 1/2) > 0$ is for the unaligned spin $j = l - 1/2$ ($p_{1/2}$, $d_{3/2}$, etc). For any single particle, the Hamiltonian $\hat{H}$ can form a complete set with $\hat{K}$, $\hat{J}^2$ and $\hat{J}_z$, i.e., ($\hat{H}$, $\hat{K}$, $\hat{J}^2$, $\hat{J}_z$). By inserting equation (19) into equation (16), we can immediately obtain two coupled ordinary differential equations for the radial parts of the Dirac eigenfunctions as follows:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right) F_{n\kappa}(r) = [M + E_{n\kappa} - V(r) + S(r)]\, G_{n\kappa}(r) \tag{20}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right) G_{n\kappa}(r) = [M - E_{n\kappa} + V(r) + S(r)]\, F_{n\kappa}(r). \tag{21}$$

By eliminating $G_{n\kappa}(r)$ in equation (20) and $F_{n\kappa}(r)$ in equation (21), we immediately obtain a second-order differential equation for the lower and upper components of the Dirac wavefunction as follows:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa - 1)}{r^2} - (M + E_{n\kappa} - \triangle(r))(M - E_{n\kappa} + \Sigma(r)) - \frac{\frac{\mathrm{d}\Sigma}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)}{M - E_{n\kappa} + \Sigma(r)}\right] G_{n\kappa}(r) = 0 \tag{22}$$

and

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa + 1)}{r^2} - (M + E_{n\kappa} - \triangle(r))(M - E_{n\kappa} + \Sigma(r)) + \frac{\frac{\mathrm{d}\triangle}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)}{M + E_{n\kappa} - \triangle(r)}\right] F_{n\kappa}(r) = 0, \tag{23}$$

where $\Sigma(r) = V(r) + S(r)$ and $\triangle(r) = V(r) - S(r)$.

It is known that the Dirac equation cannot be solved exactly for the exponential potential such as the Morse one for $\kappa \neq 0$ by using the standard methods. Therefore, an approximation has to be made: the most widely used and convenient one is the Pekeris approximation. In the next section, we introduce the Pekeris approximation.

3.2. The Pekeris approximation

The Pekeris approximation [20, 21] is based on the expansion of the centrifugal term in a series of exponentials depending on internuclear distance, keeping terms up to the second order. It should be pointed out, however, that this approximation is valid only for low vibrational energy states. In the Pekeris approximation, by change of the coordinates $x = (r - r_e)/r_e$, the centrifugal potential is expanded in a series around $x = 0$

$$V_\kappa(x) = \frac{\kappa(\kappa-1)}{r_e^2}\frac{1}{(1+x)^2} = \gamma(1 - 2x + 3x^2 - 4x^3 + \cdots), \tag{24}$$

where $\gamma = \frac{\kappa(\kappa-1)}{r_e^2}$. By taking up to the second order degrees in this series and writing them in terms of exponentials, we get

$$\widetilde{V}_\kappa(x) = \gamma(D_0 + D_1\,\mathrm{e}^{-\alpha x} + D_2\,\mathrm{e}^{-2\alpha x}). \tag{25}$$

In order to determine the constants D_0, D_1 and D_2, we also expand this potential in a series of x:

$$\widetilde{V}_\kappa(x) = \gamma\left(D_0 + D_1 + D_2 - (D_1 + 2D_2)\alpha x + \left(\frac{D_1}{2} + 2D_2\right)\alpha^2 x^2 \cdots\right). \tag{26}$$

By comparing the equal powers of equations (24) and (26), we obtain the constants D_0, D_1 and D_2 as follows:

$$D_0 = 1 - \frac{3}{\alpha} + \frac{3}{\alpha^2}, \qquad D_1 = \frac{4}{\alpha} - \frac{6}{\alpha^2}, \qquad D_2 = -\frac{1}{\alpha} + \frac{3}{\alpha^2}. \tag{27}$$

3.3. Exact pseudospin symmetry

In equation (22), we take $\triangle(r)$ as the Morse potential [22], defined as

$$V_{\mathrm{Morse}}(r) = D(\mathrm{e}^{-2\alpha x} - 2\mathrm{e}^{-\alpha x}), \qquad D > 0, \qquad \alpha > 0 \tag{28}$$

with $x = (r - r_e)/r_e$ and $\alpha = ar_e$. Here, D and α denote the dissociation energy and the Morse parameter, respectively. r_e is the equilibrium distance (bound length) between nuclei and a is a parameter to control the width of the potential well.

In the case of exact pseudospin symmetry, $\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r} = 0$. Therefore, the sum of the vector and scalar potentials $\Sigma(r)$ can be taken as a constant. If we define this constant as C, equation (22) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} - (M + E_{n\kappa} - \triangle(r))(M - E_{n\kappa} + C)\right]G_{n\kappa}(r) = 0, \tag{29}$$

where $\kappa = -\tilde{\ell}$ for $\kappa < 0$ and $\kappa = \tilde{\ell} + 1$ for $\kappa > 0$ and $\triangle(r)$ is the Morse potential. The energy eigenvalues depend on n and $\tilde{\ell}$, i.e., $E_{n\kappa} = E(n, \tilde{\ell}(\tilde{\ell}+1))$. The eigenstates with $j = \tilde{\ell} \pm 1/2$ are degenerate for $\tilde{\ell} \neq 0$. This case is commonly known as exact pseudospin symmetry [3, 13]. After inserting equations (25) and (28) into equation (29) and by using the following ansatz,

$$\nu^2 = r_e^2 D(M - E_{n\kappa} + C), \qquad \omega^2 = r_e^2\left(E_{n\kappa}^2 - M^2 - CM - CE_{n\kappa}\right). \tag{30}$$

we may easily obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - r_e^2\gamma(D_0 + D_1\,\mathrm{e}^{-\alpha x} + D_2\,\mathrm{e}^{-2\alpha x}) + \nu^2(\mathrm{e}^{-2\alpha x} - 2\,\mathrm{e}^{-\alpha x}) + \omega^2\right]G_{n\kappa}(x) = 0. \tag{31}$$

If we rewrite equation (31) by using a new variable of the form $y = \mathrm{e}^{-\alpha x}$, and by using the following ansatz

$$-\varepsilon^2 = \frac{\omega^2 - r_e^2\gamma D_0}{\alpha^2}, \qquad \beta_1^2 = \frac{-r_e^2\gamma D_1 - 2\nu^2}{\alpha^2}, \qquad \beta_2^2 = \frac{r_e^2\gamma D_2 - \nu^2}{\alpha^2}, \tag{32}$$

we obtain

$$\frac{\mathrm{d}^2 G_{n\kappa}(y)}{\mathrm{d}y^2} + \frac{1}{y}\frac{\mathrm{d}G_{n\kappa}(y)}{\mathrm{d}y} + \left[-\frac{\varepsilon^2}{y^2} + \frac{\beta_1^2}{y} - \beta_2^2\right] G_{n\kappa}(y) = 0. \tag{33}$$

In order to solve this equation with the AIM for $\kappa \neq 0$, we should transform this equation to the form of equation (1). Moreover, the wavefunction has to be the boundary conditions, i.e, $G_{n\kappa}(0) \sim y^{\varepsilon}$ for $y \to 0$ and $G_{n\kappa}(\infty) \sim \mathrm{e}^{-\beta_2 y}$ for $y \to \infty$. Therefore, the reasonable physical wavefunction we propose is as follows:

$$G_{n\kappa}(y) = y^{\varepsilon}\mathrm{e}^{-\beta_2 y} f_{n\kappa}(y). \tag{34}$$

If we insert this wavefunction into equation (33), we have the second-order homogeneous linear differential equation in the following form:

$$\frac{\mathrm{d}^2 f_{n\kappa}(y)}{\mathrm{d}y^2} = \left(\frac{2\beta_2 y - 2\varepsilon - 1}{y}\right)\frac{\mathrm{d}f_{n\kappa}(y)}{\mathrm{d}y} + \left(\frac{2\varepsilon\beta_2 + \beta_2 - \beta_1^2}{y}\right) f_{n\kappa}(y), \tag{35}$$

which is now amenable to an AIM solution. By comparing this equation with equation (1), we can write the $\lambda_0(y)$ and $s_0(y)$ values, and by means of equation (7), we may calculate $\lambda_k(y)$ and $s_k(y)$. This gives

$$\begin{aligned}
\lambda_0(y) &= \left(\frac{2\beta_2 y - 2\varepsilon - 1}{y}\right), \\
s_0(y) &= \left(\frac{2\varepsilon\beta_2 + \beta_2 - \beta_1^2}{y}\right), \\
\lambda_1(y) &= 4{\beta_2}^2 - \frac{3\beta_2(2\varepsilon+1) + {\beta_1}^2}{y} + \frac{2\varepsilon(2\varepsilon+3)+2}{y^2}, \\
s_1(y) &= \frac{2{\beta_2}^2(2\varepsilon+1) - 2{\beta_1}^2\beta_2}{y} + \frac{2{\beta_1}^2(\varepsilon+1) - 2\beta_2(3\varepsilon+1+2\varepsilon^2)}{y^2}, \\
&\ldots \text{etc.}
\end{aligned} \tag{36}$$

Combining these results with the quantization condition given by equation (14) yields

$$\begin{aligned}
s_0\lambda_1 - s_1\lambda_0 = 0 \quad &\Rightarrow \quad \varepsilon_0 = -\frac{1}{2}\frac{\beta_2 - \beta_1^2}{\beta_2}, \\
s_1\lambda_2 - s_2\lambda_1 = 0 \quad &\Rightarrow \quad \varepsilon_1 = -\frac{1}{2}\frac{3\beta_2 - \beta_1^2}{\beta_2}, \\
s_2\lambda_3 - s_3\lambda_2 = 0 \quad &\Rightarrow \quad \varepsilon_2 = -\frac{1}{2}\frac{5\beta_2 - \beta_1^2}{\beta_2}, \\
&\ldots \text{etc.}
\end{aligned} \tag{37}$$

When the above expressions are generalized, the eigenvalues turn out as

$$\varepsilon_n = \frac{\beta_1^2 - (2n+1)\beta_2}{2\beta_2}, \qquad n = 0, 1, 2, 3, \ldots \tag{38}$$

By using equations (30) and (32), we obtain the energy eigenvalues $E_{n\kappa}$,

$$2\sqrt{\gamma D_0 - (\tilde{E}_{n\kappa} - C - 2M)\tilde{E}_{n\kappa}} + \frac{2D\tilde{E}_{n\kappa} + \gamma D_1}{\sqrt{\gamma D_2 - D\tilde{E}_{n\kappa}}} + (2n+1)a = 0, \tag{39}$$

where $\tilde{E}_{n\kappa} = M + C - E_{n\kappa}$. In order to avoid getting an equation which has no solution for the energy eigenvalues, we select

$$\begin{aligned}
\sqrt{\gamma D_0 - (\tilde{E}_{n\kappa} - C - 2M)\tilde{E}_{n\kappa}} &= -\mathrm{i}\sqrt{(\tilde{E}_{n\kappa} - C - 2M)\tilde{E}_{n\kappa} - \gamma D_0}, \\
\sqrt{\gamma D_2 - D\tilde{E}_{n\kappa}} &= \mathrm{i}\sqrt{D\tilde{E}_{n\kappa} - \gamma D_2}.
\end{aligned} \tag{40}$$

Table 1. The bound state energy eigenvalues $E_{n,\kappa}$ of the Dirac particle in the rotational Morse potential for several n and κ states with $C = -10$ fm^{-1} and $C = -9.96$ fm^{-1}.

$\tilde{\ell}$	$n, \kappa < 0$	(ℓ, j)	$E_{n,\kappa<0}$ $(C = -10)$	$E_{n,\kappa<0}$ $(C = -9.96)$	$n-1, \kappa > 0$	$(\ell+2, j+1)$	$E_{n-1,\kappa>0}$ $(C = -10)$	$E_{n-1,\kappa>0}$ $(C = -9.96)$
1	$1, -1$	$(1s_{1/2})$	−0.006 4123	0.033 5874	0, 2	$(0d_{3/2})$	−0.006 4123	0.033 5874
2	$1, -2$	$(1p_{3/2})$	−0.015 5771	0.024 4201	0, 3	$(0f_{5/2})$	−0.015 5771	0.024 4201
3	$1, -3$	$(1d_{5/2})$	−0.024 3659	0.015 6261	0, 4	$(0g_{7/2})$	−0.024 3659	0.015 6261
4	$1, -4$	$(1f_{7/2})$	−0.030 5297	0.009 4547	0, 5	$(0h_{9/2})$	−0.030 5297	0.009 4547
1	$2, -1$	$(2s_{1/2})$	−0.007 0204	0.032 9795	1, 2	$(1d_{3/2})$	−0.007 0204	0.032 9795
2	$2, -2$	$(2p_{3/2})$	−0.019 0441	0.020 9547	1, 3	$(1f_{5/2})$	−0.019 0441	0.020 9547
3	$2, -3$	$(2d_{5/2})$	−0.033 7719	0.006 2238	1, 4	$(1g_{7/2})$	−0.033 7719	0.006 2238
4	$2, -4$	$(2f_{7/2})$	−0.049 2150	−0.009 2253	1, 5	$(1h_{9/2})$	−0.049 2150	−0.009 2253

Otherwise, the energy eigenvalues have the complex values. Therefore, the energy eigenvalue equation is obtained as

$$\left(2\sqrt{(\tilde{E}_{n\kappa} - C - 2M)\tilde{E}_{n\kappa} - \gamma D_0} + \frac{2D\tilde{E}_{n\kappa} + \gamma D_1}{\sqrt{D\tilde{E}_{n\kappa} - \gamma D_2}}\right)^2 + (2n+1)^2 a^2 = 0, \tag{41}$$

where $\tilde{E}_{n\kappa}$ is $M + C - E_{n\kappa}$. As a numerical example, we use the same parameters as in [11, 12] in order to compare our results with theirs. The results obtained by using the following parameters, $D = 5.0$ fm^{-1}, $r_e = 2.408\,73$ fm^{-1}, $a = 0.988\,879$ fm^{-1}, $M = 10.0$ fm^{-1}, $D_0 = 0.269\,28$, $D_1 = 0.621\,78$, $D_2 = 0.108\,93$ and $C = -10$ fm^{-1}, are shown in table 1. The pseudospin partners are clearly seen in this table, i.e., the Dirac eigenstate $1s_{1/2}$ with $n = 1$ and $\kappa = -1$ has a partner of $0d_{3/2}$ with $n - 1 = 0$ and $\kappa = 2$.

The analytical expression for the energy eigenvalue given by equation (41) and the numerical results presented in table 1 are in excellent agreement with the results of [12]. However, the results of table 2 of [12] are misleading: the authors have presented in their paper that in the case of $C = 0$, some negative bound state solutions can still exist, as shown in table 2. However, our calculations show that there are no bound states for $C \geqslant -9.95$ fm^{-1} as the numerical values are presented in table 2 for $C = -5$ fm^{-1} and $C = 0$. The authors of [12] have apparently made a simple mistake in their calculations for $C = 0$ presented in table 2. On the other hand, [11] has stated that there are no bound state solutions of the exact pseudospin symmetry Morse potential $C > -10$ fm^{-1}. But, as we have presented in table 1 for $C = -9.96$ fm^{-1}, there are negative bound state solutions and the author has also made a sign error in the eigenvalue equation. Therefore, the numerical values presented in the paper are not correct, as it is also pointed out in [12].

Hereafter, we would like to present how to find the corresponding eigenfunctions, $f_n(y)$, for the pseudospin symmetry Morse potential. By using the wavefunction generator given by equation (15), we obtain $f_n(y)$ as follows:

$$f_0(y) = 1, \tag{42}$$

$$f_1(y) = \left(2\beta_2 - \beta_1^2\right)\left(1 - \frac{2\beta_2 y}{\left(\frac{\beta 1^2 - 3\beta_2}{\beta_2} + 1\right)}\right), \tag{43}$$

$$f_2(y) = \left(\beta_1^2 - 4\beta_2\right)\left(\beta_1^2 - 3\beta_2\right)\left(1 - \frac{4\beta_2 y}{\left(\frac{\beta_1^2 - 5\beta_2}{\beta_2} + 1\right)} + \frac{4\beta_2^2 y^2}{\left(\frac{\beta_1^2 - 5\beta_2}{\beta_2} + 1\right)\left(\frac{\beta_1^2 - 5\beta_2}{\beta_2} + 2\right)}\right), \tag{44}$$

... etc.

Table 2. The same eigenvalues as in table 1 for $C = 0$ and $C = -5$ fm^{-1}.

$\tilde{\ell}$	$n, \kappa < 0$	(ℓ, j)	$E_{n,\kappa<0}$ $(C = 0)$	$E_{n,\kappa<0}$ $(C = -5)$	$n-1, \kappa > 0$	$(\ell+2, j+1)$	$E_{n-1,\kappa>0}$ $(C = 0)$	$E_{n-1,\kappa>0}$ $(C = -5)$
1	$1, -1$	$(1s_{1/2})$	9.993 5101	4.993 5463	0, 2	$(0d_{3/2})$	9.993 5101	4.993 5463
2	$1, -2$	$(1p_{3/2})$	9.983 8165	4.984 1020	0, 3	$(0f_{5/2})$	9.983 8165	4.984 1020
3	$1, -3$	$(1d_{5/2})$	9.973 7712	4.974 6641	0, 4	$(0g_{7/2})$	9.973 7712	4.974 6641
4	$1, -4$	$(1f_{7/2})$	9.965 6754	4.967 5409	0, 5	$(0h_{9/2})$	9.965 6754	4.967 5409
1	$2, -1$	$(2s_{1/2})$	9.992 9544	4.992 9662	1, 2	$(1d_{3/2})$	9.992 9544	4.992 9662
2	$2, -2$	$(2p_{3/2})$	9.980 7043	4.980 8218	1, 3	$(1f_{5/2})$	9.980 7043	4.980 8218
3	$2, -3$	$(2d_{5/2})$	9.965 2868	4.965 7281	1, 4	$(1g_{7/2})$	9.965 2868	4.965 7281
4	$2, -4$	$(2f_{7/2})$	9.948 4873	4.949 5727	1, 5	$(1h_{9/2})$	9.948 4873	4.949 5727

We can derive from the results given above the general formula for $f_n(y)$ as,

$$f_n(y) = (-1)^n \left(\prod_{k=n}^{2n-1} \left(\beta_1^2 - (k+1)\beta_2 \right) \right) {}_1F_1(-n, 2\varepsilon_n + 1; 2\beta_2 y). \tag{45}$$

Therefore, we write the total radial wavefunction as follows:

$$G_{n\kappa}(y) = (-1)^n \left(\prod_{k=n}^{2n-1} \left(\beta_1^2 - (k+1)\beta_2 \right) \right) y^{\varepsilon_n} \, \mathrm{e}^{-\beta_2 y} \, {}_1F_1(-n, 2\varepsilon_n + 1; 2\beta_2 y). \tag{46}$$

When the hypergeometric function is written in terms of the Laguerre polynomials, we obtain

$$G_{n\kappa}(y) = N y^{\varepsilon_n} \mathrm{e}^{-\beta_2 y} L_n^{2\varepsilon_n}(2\beta_2 y), \tag{47}$$

where N is the normalization constant and can be obtained from $N^2 \int_0^\infty y^{2\varepsilon_n} \, \mathrm{e}^{-2\beta_2 y} \left[L_n^{2\varepsilon_n}(2\beta_2 y) \right]^2 \mathrm{d}y = 1$ as follows:

$$N = \frac{1}{n!} (2\beta_2)^{\frac{2\varepsilon_n+1}{2}} \sqrt{\frac{(n - 2\varepsilon_n)!}{n!}}. \tag{48}$$

4. Conclusion

In this paper, we have presented the bound state solution of the Dirac equation for the attractive scalar and repulsive vector Morse potential within the framework of the asymptotic iteration method by applying the Pekeris approximation to the centrifugal-like term. We have obtained the energy eigenvalues and the corresponding eigenfunctions for the exact pseudospin symmetry case in the closed-form for any κ states. It may be seen that the (n, l, j) and $(n-1, l+2, j+1)$ states have the same energy eigenvalues as a result of exact pseudospin symmetry, as shown in tables 1 and 2. Moreover, we have also pointed out minor mistakes of previous works and have concluded that there are bound state solutions of the exact pseudospin symmetry Morse potential for $C \leqslant -9.96$ fm^{-1}, but that there are no bound state solutions of this potential $C > -9.96$ fm^{-1} by using the same potential parameters.

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